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Large-time asymptotics of a controlled large deviation probability

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1 Introduction

1.1 A large deviation control problem

- Cost-minimizing problems

$$\inf_h E[\int_0^T f(X_t, h_t)dt] \qquad \textbf{(problem on finite time horizon)}$$

$$\inf_h \overline{\lim_{T\to\infty}} \frac{1}{T} E[\int_0^T f(X_t, h_t)dt] \qquad \textbf{(problem on infinite time horizon, ergodic control)}$$

$$\begin{cases} h = (h_t)_{t\geq 0} & \text{: control} \\ X = X^h = (X_t^h)_{t\geq 0} & \text{: controlled diffusion process} \end{cases}$$

are among the classical stochastic control problems (cf. Fleming & Soner).

- Corresponding to the above problems, we can consider the problem of maximizing the probability

$$P\Big(\frac{1}{T}\int_0^T f(X_t, h_t)dt \leq k\Big)$$

over a large time interval, for a given level $k \in \mathbb{R}$.

This is a non-conventional stochastic control problem (The Dynamic Programming Principle is not directly applicable).

• According to the idea of Large deviations (e.g. Gärtner-Ellis theorem), we expect that, if the limit

$$\Lambda(\theta) = \lim_{T\to\infty} \frac{1}{T} \sup_h \log E[e^{\theta \int_0^T f(X_t,h_t)dt}] \tag{1}$$

exists and some regulality properties hold for $\Lambda(\theta)$, then the behaviour of the maximized probability as $T \to \infty$ is like

$$\frac{1}{T} \sup_h \log P\Big(\frac{1}{T}\int_0^T f(X_t,h_t)dt \le k\Big) \approx -\inf_{k'\in(-\infty,k]} I(k'),$$

where the "rate function" $I(k)$ is given by the Legendre transform of $\Lambda(\theta)$:

$$I(k) = \sup_\theta \{k\theta - \Lambda(\theta)\}.$$

■Formal argument to obtain the upper bound:

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \sup_h \log P\Big(\frac{1}{T}\int_0^T f(X_t,h_t)dt \le k\Big) \le - \inf_{k'\in(-\infty,k]} I(k').$$

By Chebyshev's inequality, for any $\theta \in (-\infty, 0)$,

$$\begin{aligned} E[e^{\theta\int_0^T f(X_t,h_t)dt}] &\ge E\Big[e^{\theta\int_0^T f(X_t,h_t)dt} \quad : \quad \frac{1}{T}\int_0^T f(X_t,h_t)dt \le k\Big] \\ &\ge e^{\theta kT} P\Big(\frac{1}{T}\int_0^T f(X_t,h_t)dt \le k\Big), \end{aligned}$$

and so

$$\log P\Big(\frac{1}{T}\int_0^T f(X_t,h_t)dt \le k\Big) \le \log E[e^{\theta\int_0^T f(X_t,h_t)dt}] - \theta k.$$

Hence

$$\begin{aligned} \overline{\lim_{T\to\infty}} \frac{1}{T} \sup_h \log P\Big(\frac{1}{T}\int_0^T f(X_t,h_t)dt \le k\Big) &\le \overline{\lim_{T\to\infty}} \frac{1}{T} \sup_h \log E[e^{\theta\int_0^T f(X_t,h_t)dt}] - \theta k \\ &= \Lambda(\theta) - \theta k. \end{aligned}$$

Therefore

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \sup_h \log P\Big(\frac{1}{T}\int_0^T f(X_t, h_t)dt \le k\Big) \le \inf_{\theta\in(-\infty,0)} \{\Lambda(\theta) - \theta k\}.$$

Hence, if we define $I(k) := \sup_{\theta\in(-\infty,0)} \{k\theta - \Lambda(\theta)\}$, $I(k)$ is non-increasing and

$$\begin{aligned}\overline{\lim_{T\to\infty}} \frac{1}{T} \sup_h \log P\Big(\frac{1}{T}\int_0^T f(X_t, h_t)dt \le k\Big) &\le -I(k) \\ &= -\inf_{k'\in(-\infty,k]} I(k').\end{aligned}$$

■The key to prove the lower bound:

$$\underline{\lim}_{T\to\infty} \frac{1}{T} \sup_h \log P\Big(\frac{1}{T}\int_0^T f(X_t, h_t)dt \le k\Big) \ge -\inf_{k'\in(-\infty,k]} I(k').$$

- If $\Lambda(\theta)$ convex (which is formally true) and C^1, an explicit expression for $I(k)$ is possible for $k \in$ Range$\{\Lambda'(\theta) : \theta \in (-\infty, 0)\}$, as in the proof of Gärtner-Ellis theorem.

- It is enough to show that

$$\underline{\lim}_{T\to\infty} \frac{1}{T} \log P\Big(\frac{1}{T}\int_0^T f(X_t, \hat{h}_t)dt \le k\Big) \ge -\inf_{k'\in(-\infty,k]} I(k') \qquad (X = X^{\hat{h}})$$

for a suitable $\hat{h}$ (which may depend on k). How do we choose $\hat{h}$ and a measure transformation?

1.2 Related studies in the context of math. finance

Upside chance maximization Pham (2003), Hata&Sekine (2005, 2010), etc.

Downside risk minimization Hata-Nagai-Sheu (2010), etc.

- An example of downside risk minimization problem:

$$\text{Security prices: } \begin{cases} dS_t^0 = r(X_t)S_t^0 dt, \\ dS_t^i = S_t^i\{\alpha^i(X_t)dt + \sum_{i=1}^{m+n} \sigma_k^i(X_t)dW_t^k\}, \quad i = 1, \ldots, m. \end{cases}$$

$$\text{Economic factors: } dX_t^j = \beta^j(X_t)dt + \sum_{k=1}^{m+n} \lambda_k^j(X_t)dW_t^k, \quad j = 1, \ldots, n.$$

$$\text{Wealth process: } V_t = V_t(h), \qquad \frac{dV_t}{V_t} = \sum_{i=0}^{m} h_t^i \frac{dS_t^i}{S_t^i}$$

Then

$$\log \frac{V_T(h)}{S_T^0} = \int_0^T \Big\{ h_t^* (\alpha(X_t) - r(X_t)\mathbf{1}) - \frac{1}{2}|\sigma^*(X_t)h_t|^2 \Big\} dt + \int_0^T h_t^* \sigma(X_t) dW_t.$$

One tries to prove

$$\lim_{T\to\infty} \frac{1}{T} \inf_h \log P\Big(\frac{1}{T} \log \frac{V_T(h)}{S_T^0} \le k \Big) = - \inf_{k' \in (-\infty, k]} I(k'),$$

where

$$I(k) = \sup_{\gamma \in (-\infty, 0)} \{ k\gamma - \chi(\gamma) \},$$

$$\chi(\gamma) = \lim_{T\to\infty} \frac{1}{T} \inf_h \log E\Big[\Big(\frac{V_T(h)}{S_T^0} \Big)^\gamma \Big].$$

• We are going to prove an analogous statement without using a specific structure of financial market models.

1.3 Preliminaries

- $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\in[0,\infty)})$: a filtered prob. space
- $B = (B_t)_{t\in[0,\infty)}$: a standard $\mathcal{F}_t$-Brownian motion in $\mathbb{R}^d$.
- We consider the controlled SDE

$$\begin{cases} dX_t = \sigma(X_t)dB_t + \big[\beta(X_t) + \gamma(X_t)h_t\big]dt \\ X_0 = x \in \mathbb{R}^n \end{cases} \tag{2}$$

where the coefficients satisfy

$$\begin{cases} \sigma(x) \in C_b^1(\mathbb{R}^n; \mathbb{R}^n \times \mathbb{R}^d), \\ \beta(x) \in C^1(\mathbb{R}^n; \mathbb{R}^n), \qquad |\nabla\beta(x)| \le C, \\ \gamma(x) \in C_b^1(\mathbb{R}^n; \mathbb{R}^n \times \mathbb{R}^m). \end{cases}$$

We also assume

$$\exists \nu_1, \nu_2 > 0, \quad \forall x, \xi \in \mathbb{R}^n, \quad \nu_1|\xi|^2 \le \sigma\sigma^*(x)\xi\cdot\xi \le \nu_2|\xi|^2.$$

- For each $T \in (0,\infty)$, the totality of $\mathbb{R}^k$-valued $\mathcal{F}_t$-progressively measurable processes $(z_t)_{t\in[0,T]}$ such that $P\Big(\int_0^T |z_t|^2 dt < \infty\Big) = 1$ is denoted by $\mathbf{L}^2[0,T]^k$.

- **(Admissible controls)** For each $T \in (0, \infty)$ and $x \in \mathbb{R}^n$, we define

$$\mathcal{A}(T,x) := \Big\{ (h_t)_{t\in[0,T]} \in \mathbf{L}^2[0,T]^m \ \Big| \ \text{the solution } X = (X_t)_{t\in[0,T]} \text{ of the SDE (2) uniquely exists in } \mathbf{L}^2[0,T]^n \text{ and does not explode in } [0,T] \Big\}$$

- Cost functional:

$$\begin{aligned} f(x,h) &= V(x) + \frac{1}{2} S(x) h \cdot h + g(x) \cdot h \\ &= \underbrace{V(x) - \frac{1}{2} S^{-1} g \cdot g(x)}_{=:U(x)} + \frac{1}{2} S(x) \Big(h + S^{-1} g(x) \Big) \cdot \Big(h + S^{-1} g(x) \Big) \end{aligned}$$

$V(x) \in C^1(\mathbb{R}^n)$, $|\nabla V(x)| \le C(|x|+1)$,
$S(x) \in C_b^1(\mathbb{R}^n; \mathbb{R}^m \times \mathbb{R}^m)$: symmetric and (strictly) positive definite for each $x \in \mathbb{R}^n$,
$g(x) \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, $|g(x)| \le C(|x|+1)$.

- **Main assumption:**

$$U(x) := V(x) - \frac{1}{2}S^{-1}g \cdot g(x) \geq c_U|x|^2 - c'_U$$

for some $c_U > 0$ and $c'_U \in \mathbb{R}$.

- For each $T \in (0, \infty)$, $x \in \mathbb{R}^n$ and $\theta \in (-\infty, 0)$, define

$$J(T; x; \theta) := \sup_{h \in \mathcal{A}(T;x)} \log E[e^{\theta \int_0^T f(X_t, h_t) dt}]. \tag{3}$$

By the Dynamic Programming Principle, the function $v(t, x) = J(T - t; x; \theta)$ should formally satisfy the following HJB equation

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\sigma\sigma^*(x)D^2 v] + \frac{1}{2}|\sigma^*(x)\nabla v|^2 + \beta(x) \cdot \nabla v + \sup_h \{\gamma(x)h \cdot \nabla v + \theta f(x, h)\} \\ = 0 \quad \text{in } [0, T) \times \mathbb{R}^n \\ v = 0 \quad \text{on } \{t = T\} \times \mathbb{R}^n \end{cases}$$

In our setting, we can calculate $\sup\{\cdots\}$, and the maximizer is

$$\hat{h} = -S^{-1}\Big(\frac{1}{\theta}\gamma^* \nabla v + g\Big).$$

The equation is rewritten as

$$\left\{ \begin{array}{ll} \frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\sigma\sigma^*(x)D^2 v] + \frac{1}{2}N_\theta(x)\nabla v \cdot \nabla v + G(x)\cdot \nabla v + \theta U(x) = 0 & \text{in } [0,T)\times \mathbb{R}^n \\ v(T,\cdot) = 0 & \text{on } \{t = T\}\times \mathbb{R}^n \end{array} \right. \tag{4}$$

where

$$N_\theta(x) := \sigma\sigma^*(x) - \frac{1}{\theta}\gamma S^{-1}\gamma^*(x),$$

$$G(x) := \beta(x) - \gamma S^{-1} g(x),$$

$$U(x) := V(x) - \frac{1}{2} g^* S^{-1} g(x).$$

1.4 Main result and outline of proof

- Thanks to the previous studies (by Bensoussan, Frehse, Nagai, Ichihara, Sheu), our HJB equation (4) turns out to have a classical solution $v(t,x) = v(t,x;T;\theta)$. By performing some estimation for the solution, we can prove the verification theorem:

$$v(0,x;T;\theta) = \sup_{h \in \mathcal{A}(x;T)} \log E[e^{\theta \int_0^T f(X_t,h_t)dt}].$$

- Definition of $\Lambda(\theta)$. Instead of defining $\Lambda(\theta)$ as the limit

$$\lim_{T\to\infty} \frac{1}{T} \sup_{h \in \mathcal{A}(T;x)} \log E[e^{\theta \int_0^T f(X_t,h_t)dt}]$$

directly, we consider the ergodic-type HJB (EHJB) equation

$$\Lambda = \frac{1}{2}\mathrm{tr}[\sigma\sigma^*(x)D^2 w] + \frac{1}{2}N_\theta(x)\nabla w \cdot \nabla w + G(x)\cdot\nabla w + \theta U(x). \tag{5}$$

The structure theorem (by Kaise, Sheu, Ichihara) for EHJB equations tells us that there is a unique "bottom" solution $(\Lambda^*, w^*(x))$. We define $\Lambda(\theta) := \Lambda^*$, $w_\theta(x) := w_*(x)$, and verify that

$$\Lambda(\theta) = \lim_{T\to\infty} \frac{1}{T} \sup_{h \in \mathcal{A}(T;x)} \log E[e^{\theta \int_0^T f(X_t,h_t)dt}] \qquad (\forall x \in \mathbb{R}^n).$$

• Regularity of $\Lambda(\theta)$. Through the analysis of the EHJB equation (5) w.r.t. θ, we can prove that $\Lambda(\theta)$ and $w_\theta(x)$ are C^1 w.r.t. θ and the derivatives $\Lambda'(\theta)$, w'_θ satisfy the Poisson equation

$$\Lambda'(\theta) = L_\theta w'_\theta + V_\theta(x),$$

where

$$L_\theta = \frac{1}{2}\mathrm{tr}[\sigma\sigma^* D^2] + \left[G + N_\theta \nabla w_\theta\right] \cdot \nabla,$$

$$V_\theta(x) := \frac{1}{2} N'_\theta \nabla w_\theta \cdot \nabla w_\theta(x) + U(x).$$

Theorem 1 (Main Theorem). *(i) $\Lambda(\theta)$ is a C^1, convex function of $\theta \in (-\infty, 0)$.*
(ii) If $k \in (\Lambda'(-\infty), \Lambda'(0-))$, the limit

$$\Pi(k) := \lim_{T\to\infty} \frac{1}{T} \sup_{h\in\mathcal{A}(T;x)} \log P\Big(\frac{1}{T}\int_0^T f(X_t, h_t)dt \leq k\Big)$$

exists and

$$\Pi(k) = -\inf_{k'\in(-\infty,k]} I(k') = -I(k).$$

where

$$I(k) := \inf_{\theta\in(-\infty,0)} \{k\theta - \Lambda(\theta)\}.$$

2 HJB equation and the verification theorem

2.1 Existence, uniqueness of solutions

Our HJB equation is

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\sigma\sigma^*(x)D^2 v] + \frac{1}{2}N_\theta(x)\nabla v \cdot \nabla v + G(x)\cdot\nabla v + \theta U(x) = 0 & \text{in } [0,T)\times\mathbb{R}^n \\ v(T,\cdot) = 0 & \text{on } \{t=T\}\times\mathbb{R}^n. \end{cases} \tag{6}$$

Instead of (6) we first consider the Cauchy problem

$$\begin{cases} \frac{\partial \bar{v}}{\partial t} - \frac{1}{2}\mathrm{tr}[\sigma\sigma^*(x)D^2 \bar{v}] + \frac{1}{2}N_\theta(x)\nabla \bar{v} \cdot \nabla \bar{v} - G(x)\cdot\nabla \bar{v} + \theta U(x) = 0 & \text{in } (0,\infty)\times\mathbb{R}^n \\ \bar{v} = 0 & \text{on } \{t=0\}\times\mathbb{R}^n \end{cases} \tag{7}$$

(Afterwards, setting $v(t,x;T) := -\bar{v}(T-t,x)$, we obtain the solution of our HJB equation (4).)

The equation can be rewritten as

$$\frac{\partial \bar{v}}{\partial t} - \frac{1}{2}\mathrm{tr}[\sigma\sigma^*(x)D^2\bar{v}] - G(x)\cdot\nabla\bar{v} + \inf_{\xi\in\mathbb{R}^n}\{-\xi\cdot\nabla\bar{v} + L(x,\xi)\} = 0,$$

where

$$L(x,\xi) = \frac{1}{2}N_\theta^{-1}(x)\xi\cdot\xi - \theta U(x).$$

Theorem 2 (Nagai(1996), Bensoussan-Frehse-Nagai(1998), Ichihara-Sheu(2011)**).** *There exists a unique solution* $\bar{v}(t,x) \in C^{1,2}((0,\infty)\times\mathbb{R}^n)\cap C([0,\infty)\times\mathbb{R}^n)$ *of (7) such that* $\inf_{0\le t\le T}\inf_{x\in\mathbb{R}^n}\bar{v}(t,x) > -\infty$ *for each* $T\in(0,\infty)$*. The solution admits a stochastic representation*

$$\bar{v}(T,x) = \inf_{\xi\in\tilde{\mathcal{A}}(T;x)} E[\int_0^T L(X_t,\xi_t)dt] \tag{8}$$

where

$$dX_t = \sigma(X_t)dB_t + [G(X_t)-\xi_t]dt, \;\; 0\le t\le T, \;\; X_0 = x\in\mathbb{R}^n.$$

- Uniqueness is a consequence of the stochastic representation (8).
- Existence follows from purely PDE theoretic arguments.

2.2 Verification theorem

Theorem 3. *For the solution of the HJB equation* $v(t,x) = v(t,x;T) := -\bar{v}(T-t,x)$*, set*

$$\hat{h}(t,x;T) := -S^{-1}(x)\Big(\frac{1}{\theta}\gamma^*(x)\nabla v(t,x;T) + g(x)\Big).$$

Let $\hat{X} = (\hat{X})_{0\le t\le T}$ *be the solution of the s.d.e.*

$$\begin{cases} d\hat{X}_t = \sigma(\hat{X}_t)dB_t + \big[\beta(\hat{X}_t) + \gamma(\hat{X}_t)\hat{h}(t,\hat{X}_t)\big]dt \\ \hat{X}_0 = x \in \mathbb{R}^n \end{cases}$$

Assume that

$$\begin{cases} \hat{X} \text{ is non-explosive on } [0,T], \\ E[e^{\int_0^T [\sigma^*\nabla v(t,\hat{X}_t)]^* dB_t - \frac{1}{2}\int_0^T |\sigma^*\nabla v(t,\hat{X}_t)|^2 dt}] = 1. \end{cases} \tag{9}$$

Then

$$\begin{aligned} v(0,x;T) &= \log E[e^{\theta\int_0^T f(\hat{X}_t,\hat{h}(t,\hat{X}_t)dt}] \\ &= \sup_{h\in\mathcal{A}(T;x)} \log E[e^{\theta\int_0^T f(\hat{X}_t,\hat{h}(t,\hat{X}_t)dt}]. \end{aligned}$$

2.3 Estimation of the solution

Lemma 2.1. *The solution $v(t,x) = v(t,x;T)$ of the HJB equation (4) satisfies*

$$\frac{\partial v}{\partial t} \geq -K,$$

where

$$-K := \inf_{x \in \mathbb{R}^n} -\theta U(x).$$

Moreover, for each $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$, $c > 0$ and $\rho > 0$, the following estimation holds:

$$\begin{aligned}
&|\nabla v(t_0,x_0)|^2 + \frac{4(1+c)}{\hat{\nu}_1}\Big(\frac{\partial v}{\partial t}(t_0,x_0) + K\Big) \\
&\leq C\Big(||N_\theta||^2_{L^\infty(B_\rho(x_0))} + ||\nabla N_\theta||^2_{L^\infty(B_\rho(x_0))} + ||\nabla \sigma\sigma^*||^2_{L^\infty(B_\rho(x_0))} + ||\beta||^2_{L^\infty(B_\rho(x_0))} \\
&\qquad + ||\nabla \beta||_{L^\infty(B_\rho(x_0))} + ||U||_{L^\infty(B_\rho(x_0))} + ||\nabla U||_{L^\infty(B_\rho(x_0))} + 1\Big),
\end{aligned}$$

where $C > 0$ is a positive constant depends only on n, c, ρ, K, ν_1, ν_2 and $\hat{\nu}_1 = \underline{\lambda}(N_\theta)$.

Lemma 2.2. *Let W_t be an m-dimensional $\mathcal{F}_t$-Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ and $T \in (0, \infty)$. We consider the SDE*

$$dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt, \qquad 0 \le t \le T, \qquad X_0 = x \in \mathbb{R}^n. \tag{10}$$

Here we assume that the functions $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ are Borel measurable and locally Lipschitz w.r.t. the spatial variable. Set

$$L := \frac{\partial\psi}{\partial t} + \frac{1}{2}\mathrm{tr}[\sigma\sigma^*(t,x)D^2] + b(t,x)\cdot\nabla$$

and let $\mathbf{a} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$ be a Borel measurable function. Suppose that there exist a positive function $\psi \in C^{1,2}([0,T] \times \mathbb{R}^n)$ and a constant $C = C_T > 0$ such that

$$\lim_{\rho\to\infty} \inf_{t\in[0,T]} \inf_{|x|\ge\rho} \psi(t,x) = \infty, \tag{11}$$

$$L\psi \le C\psi, \tag{12}$$

$$|\mathbf{a}|^2 \le C\psi, \tag{13}$$

$$|\sigma^*\nabla\psi| \le C\psi, \tag{14}$$

$$L\psi + \mathbf{a}\cdot\sigma^*\nabla\psi \le C\psi. \tag{15}$$

Then, the SDE (10) has a unique non-explosive solution X_t on $[0,T]$ and it satisfies

$$E\left[\exp\left\{\int_0^t \mathbf{a}^*(s,X_s)dW_s - \frac{1}{2}\int_0^t |\mathbf{a}(s,X_s)|^2 ds\right\}\right] = 1 \tag{16}$$

for all $t \in [0,T]$.

3 Analysis of the EHJB equation (w.r.t. the parameter θ)

We consider the EHJB equation

$$\Lambda = \frac{1}{2}\mathrm{tr}[\sigma\sigma^*(x)D^2 w] + \frac{1}{2}N_\theta(x)\nabla w \cdot \nabla w + G(x)\cdot\nabla w + \theta U(x), \tag{17}$$

which is the equation for the problem

$$\sup_h \lim_{T\to\infty}\frac{1}{T}\log E[e^{\theta\int_0^T f(X_t,h_t)dt}], \qquad \theta \in (-\infty, 0), \tag{18}$$

3.1 The structure of EHJB equations (Kaise & Sheu 2006)

$$\Lambda = \frac{1}{2}\mathrm{tr}[a(x)D^2 w] + \frac{1}{2}\hat{a}(x)\nabla w\cdot\nabla w + b(x)\cdot\nabla w + V(x) \qquad x\in\mathbb{R}^n \tag{19}$$

Assumptions:

(ks1) a^{ij}, $\hat{a}^{ij}$, b^i, V : sufficiently smooth

(ks2) $a, \hat{a}$: uniformly positive definite

(ks3) $\exists\Psi\in C^2(\mathbb{R}^n)$ s.t.

$$\frac{1}{2}\mathrm{tr}[aD^2\Psi] + \frac{1}{2}\hat{a}\nabla\Psi\cdot\nabla\Psi + b\cdot\nabla\Psi + V \to -\infty \quad \text{as} \quad |x|\to\infty.$$

Set

$$\mathcal{A} := \{\Lambda \in \mathbb{R} : \text{there exists a smooth function } w \text{ satisfying (19) for } \Lambda\ \},$$

$$Lf := \frac{1}{2}\mathrm{tr}[aD^2 f] + [b + \hat{a}\nabla w] \cdot \nabla f, \qquad f \in C^2(\mathbb{R}^n).$$

Theorem 4. *(Kaise-Sheu (2006)) There is a number $\Lambda^* \in \mathbb{R}$ such that*

$$\mathcal{A} = [\Lambda^*, +\infty),$$

and the following dichotomy holds:

$$\Lambda > \Lambda^* \Longrightarrow L \ \textit{ is transient},$$
$$\Lambda = \Lambda^* \Longrightarrow L \ \textit{ is ergodic (positive recurrent)}.$$

Moreover, the solution w corresponding to the bottom Λ^* is unique up to additive constants.

• We define $\Lambda(\theta), w_\theta(x)$ as the "bottom" of our EHJB equation (5). The corresponding operator

$$L_\theta = \frac{1}{2}\mathrm{tr}[\sigma\sigma^*(x)D^2] + [G(x) + N_\theta \nabla w_\theta] \cdot \nabla$$

is ergodic.

• A direct proof that L_θ is ergodic is possible. By an argument based on a maximum principle we can show that

$$w_\theta(x) \to -\infty \quad \text{as} \quad |x| \to \infty$$

Then $-w_\theta$ turns out to be a Lyapunov function of L_θ.

3.2 Convergence of the solution of HJB equation

Theorem 5.

$$\lim_{T\to\infty} \frac{v(0, x; T; \theta)}{T} = \Lambda(\theta)$$

.

• Ichihara & Sheu (2011) proved a stronger statement.

3.3 Convexity of $\Lambda(\theta)$

By the verification theorem and the previous theorem we have

$$\Lambda(\theta) = \lim_{T\to\infty} \frac{1}{T} \sup_{h\in\mathcal{A}(T;x)} \log E[e^{\theta \int_0^T f(X_t,h_t)dt}].$$

Convexity follows from this formula .

3.4 Differentiability of $\Lambda(\theta)$

Differentiating the EHJB equation w.r.t. θ formally, we have

$$\begin{aligned}\Lambda'(\theta) &= \frac{1}{2}\mathrm{tr}[\sigma\sigma^* D^2 w'] + N_\theta \nabla w \cdot \nabla w' + \frac{1}{2} N'_\theta \nabla w \cdot \nabla w + G(x)\cdot \nabla w' + U \\ &= \frac{1}{2}\mathrm{tr}[\sigma\sigma^* D^2 w'] + (G(x) + N_\theta \nabla w)\cdot \nabla w' + \underbrace{\frac{1}{2} N'_\theta \nabla w \cdot \nabla w + U}_{=:V_\theta(x)} .\end{aligned}$$

Hence we expect that the pair $(\Lambda'(\theta), w')$ is a solution of the Poisson equation

$$\Lambda'(\theta) = L_\theta w' + V_\theta(x), \tag{20}$$

where

$$V_\theta(x) := \frac{1}{2} N'_\theta \nabla w_\theta \cdot \nabla w_\theta(x) + U(x) = \frac{1}{2\theta^2} \gamma S^{-1} \gamma^* \nabla w_\theta \cdot \nabla w_\theta(x) + U(x).$$

Moreover, we can formally write

$$\Lambda'(\theta) = \int_{\mathbb{R}^n} V_\theta(x) d\mu_\theta(x),$$

where μ_θ is the invariant distribution of L_θ.

• These formal arguments are justified by the following result (which can be proved using the ideas of Bensoussan).

- We consider an operator

$$L = \frac{1}{2}\mathrm{tr}[a(x)D^2] + b(x)\cdot\nabla = \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(x)D_{ij} + \sum_{i=1}^{n} b^i(x)D_i, \qquad x \in \mathbb{R}^n$$

and a function $f \in C^\infty(\mathbb{R}^n)$, satisfying the following assumptions:
(i) $a^{ij}(x)$, $b^i(x)$, $i,j = 1,\ldots,n$, belong to $C^\infty(\mathbb{R}^n)$ and $a(x) = [a^{ij}(x)]_{i,j}$ is uniformly nondegenerate:

$$\exists\nu > 0,\ \forall x,\xi \in \mathbb{R}^n,\ a^{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2.$$

(ii) there exist a number $R_0 > 0$, a function $\psi \in C^2(\mathbb{R}^n \rightsquigarrow (0,\infty))$, and a constant $c > 0$ such that

$$\lim_{R\to\infty}\inf_{x\in B_R^c}\psi(x) = \infty, \tag{21}$$

$$L\psi < -1 \quad \text{outside } \ B_{R_0},$$

$$L\psi + \frac{c}{\psi}a\nabla\psi\cdot\nabla\psi < 0 \quad \text{outside } \ B_{R_0}.$$

(iii) f satisfies $\sup_{x\in B_{R_0}^c}\frac{|f(x)|}{-L\psi(x)} < \infty$.
Let $m = m(dx)$ be the invariant distribution of L.

Theorem 6. *The linear problem*

$$\begin{cases} -Lz = f \quad \textit{in } \mathbb{R}^n, \\ z \in C^\infty(\mathbb{R}^n), \quad \sup_{x \in B_{R_0}^c} \frac{|z(x)|}{\psi(x)} < \infty \end{cases} \tag{22}$$

is solvable if and only if

$$\int_{\mathbb{R}^n} f(x) m(dx) = 0.$$

Furthermore, the function z that satisfies (22) is uniquely determined up to an additive constant.

- Using the ideas of the proof of the theorem, we can prove

Theorem 7. $\Lambda(\theta)$, $\theta \in (-\infty, 0)$, *is a* C^1 *function. Moreover,* $\Lambda'(\theta)$ *has an expression*

$$\begin{aligned}\Lambda'(\theta) &= \int_{\mathbb{R}^n} V_\theta(x) d\mu_\theta(x) \\ &= \int_{\mathbb{R}^n} \frac{1}{2\theta^2} \gamma S^{-1} \gamma^* \nabla w_\theta \cdot \nabla w_\theta(x) d\mu_\theta(x),\end{aligned}$$

where μ_θ *is the invariant distribution of* L_θ*.*

4 Proof of the main result

4.1 Upper bound

For $\theta \in (-\infty, 0)$,

$$E[e^{\theta \int_0^T f(X_t,h_t)dt}] \geq E\Big[e^{\theta \int_0^T f(X_t,h_t)dt} \quad : \quad \frac{1}{T}\int_0^T f(X_t,h_t)dt \leq k\Big]$$
$$\geq e^{\theta k T} P\Big(\frac{1}{T}\int_0^T f(X_t,h_t)dt \leq k\Big).$$

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \sup_{h\in\mathcal{A}(T;x)} \log P\Big(\frac{1}{T}\int_0^T f(X_t,h_t)dt \leq k\Big)$$
$$\leq \overline{\lim_{T\to\infty}} \frac{1}{T} \sup_{h\in\mathcal{A}(T;x)} \log E[e^{\theta \int_0^T f(X_t,h_t)dt}] - \theta k$$
$$= \Lambda(\theta) - \theta k.$$

It follows that

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \sup_{h\in\mathcal{A}(T;x)} \log P\Big(\frac{1}{T}\int_0^T f(X_t,h_t)dt \leq k\Big) \leq -I(k).$$

4.2 Lower bound

Let $\Lambda'(-\infty) < k - \epsilon < k < \Lambda'(0-)$. Since $I(\cdot)$ is continuous, it is enough to prove that

$$\varliminf_{T\to\infty} \frac{1}{T} \sup_{h\in\mathcal{A}(T;x)} \log P\Big(\frac{1}{T}\int_0^T f(X_t, h_t)dt \le k\Big) \ge -I(k-\epsilon) - 2\epsilon. \tag{23}$$

We can choose $\theta_* = \theta_*(k, \epsilon) \in (-\infty, 0)$ such that

$$\Lambda'(\theta_*) = k - \epsilon.$$

Then, since $\Lambda(\cdot)$ is convex, we have, for any $\theta \in (-\infty, 0)$,

$$\Lambda(\theta) \ge \Lambda(\theta_*) + \Lambda'(\theta_*)(\theta - \theta_*) = \Lambda(\theta_*) + (k - \epsilon)(\theta - \theta_*).$$

Namely,

$$\theta_*(k - \epsilon) - \Lambda(\theta_*) \ge \theta(k - \epsilon) - \Lambda(\theta).$$

Therefore

$$I(k - \epsilon) = \theta_*(k - \epsilon) - \Lambda(\theta_*) = \theta_*\Lambda'(\theta_*) - \Lambda(\theta_*).$$

The inequality (23) can be rewritten as

$$\varliminf_{T\to\infty} \frac{1}{T} \sup_{h\in\mathcal{A}(T;x)} \log P\Big(\frac{1}{T}\int_0^T f(X_t,h_t)dt \le \Lambda'(\theta_*)+\epsilon\Big) \ge \Lambda(\theta_*) - \theta_*\Lambda'(\theta_*) - 2\epsilon.$$

Let

$$\hat{h}(x) := -S^{-1}(x)\Big(\frac{1}{\theta_*}\gamma^*(x)\nabla w(x) + g(x)\Big)$$

and $\hat{X}_t$ be the solution of the s.d.e.

$$\begin{cases} d\hat{X}_t = \sigma(\hat{X}_t)dB_t + \big[\beta(\hat{X}_t) + \gamma(\hat{X}_t)\hat{h}(\hat{X}_t)\big]dt \\ \hat{X}_0 = x \in \mathbb{R}^n. \end{cases}$$

This is an optimal controlled process for the problem (18) with $\theta = \theta_*$. It is enough to prove the inequality

$$\varliminf_{T\to\infty} \frac{1}{T} \log P\Big(\frac{1}{T}\int_0^T f(\hat{X}_t,\hat{h}_t)dt \le \Lambda'(\theta_*)+\epsilon\Big) \ge \Lambda(\theta_*) - \theta_*\Lambda'(\theta_*) - 2\epsilon.$$

Define a probability measure $\hat{P}$ by

$$\frac{d\hat{P}}{dP}\bigg|_{\mathcal{F}_T} = e^{\int_0^T [\sigma^*\nabla w(\hat{X}_t)]^* dB_t - \frac{1}{2}\int_0^T |\sigma^*\nabla w(\hat{X}_t)|^2 dt}.$$

Then

$$\hat{B}_t := B_t + \int_0^t \sigma^* \nabla w(\hat{X}_s) ds$$

is a Brownian motion under $\hat{P}$. The dynamics of $\hat{X}_t$ can be rewritten as

$$\begin{aligned} d\hat{X}_t &= \sigma(\hat{X}_t) d\hat{B}_t + (\beta + \gamma \hat{h} + \sigma\sigma^* \nabla w)(\hat{X}_t) dt \\ &= \sigma(\hat{X}_t) d\hat{B}_t + (G + N_{\theta_*} \nabla w)(\hat{X}_t) dt. \end{aligned}$$

Hence $\hat{X}_t$ is an L_{θ_*}-diffusion under $\hat{P}$. Write

$$\hat{M}_T := \int_0^T [\sigma^* \nabla w(\hat{X}_t)]^* d\hat{B}_t,$$

and define events A_i, $i = 0, 1, 2$, by

$$\begin{aligned} A_0 &:= \Big\{ \frac{1}{T} \int_0^T f(\hat{X}_t, \hat{h}(\hat{X}_t)) dt \leq \Lambda'(\theta_*) + \epsilon \Big\}, \\ A_1 &:= \Big\{ -\hat{M}_T \geq -\epsilon T \Big\}, \\ A_2 &:= \Big\{ -\frac{1}{2} \langle \hat{M} \rangle_T \geq (\Lambda(\theta_*) - \theta_* \Lambda'(\theta_*) - \epsilon) T \Big\}. \end{aligned}$$

Then

$$
\begin{aligned}
P(A_0) &= \hat{E}[e^{-\hat{M}_T - \frac{1}{2}\langle \hat{M} \rangle_T} : A_0] \\
&\geq \hat{E}[e^{-\hat{M}_T - \frac{1}{2}\langle \hat{M} \rangle_T} : A_0 \cap A_1 \cap A_2] \\
&\geq e^{(\Lambda(\theta_*) - \theta_* \Lambda'(\theta_*) - 2\epsilon)T} \hat{P}(A_0 \cap A_1 \cap A_2) \\
&\geq e^{(\Lambda(\theta_*) - \theta_* \Lambda'(\theta_*) - 2\epsilon)T} (1 - \hat{P}(A_0^c) - \hat{P}(A_1^c) - \hat{P}(A_2^c)).
\end{aligned}
\tag{24}
$$

We can prove that

$$
\hat{P}(A_i^c) \leq \frac{C_i(\epsilon)}{T}, \quad i = 0, 1, 2,
$$

for some positive constants $C_i(\epsilon)$ which depend on ϵ but are independent of T. It follows that

$$
\underline{\lim}_{T \to \infty} \frac{1}{T} \log P(A_0) \geq \Lambda(\theta_*) - \theta_* \Lambda'(\theta_*) - 2\epsilon.
$$

5 Conclusion and comments

• We can formulate a large deviation control problem and prove (a simple form of) a large deviation principle (in a particular case).

• We can also prove

$$\sup_{h \in \mathcal{A}(x)} \lim_{T\to\infty} \frac{1}{T} \log P\Big(\frac{1}{T}\int_0^T f(X_t, h_t)dt \le k\Big) = -\inf_{k' \in (-\infty, k]} I(k').$$

(In this case the definition of $\mathcal{A}(x)$ is more complicated.)

• It seems interesting to consider the problem of minimizing the probability

$$P\Big(\frac{1}{T}\int_0^T f(X_t, h_t)dt \ge k\Big),$$

$$\Lambda(\theta) = \lim_{T\to\infty} \frac{1}{T} \inf_h \log E[e^{\theta \int_0^T f(X_t, h_t)dt}],$$

$$\frac{1}{T} \inf_h \log P\Big(\frac{1}{T}\int_0^T f(X_t, h_t)dt \ge k\Big) \approx -\inf_{k' \in [k, \infty)} I(k'),$$

$$I(k) = \sup_\theta \{k\theta - \Lambda(\theta)\}.$$